NORTH-HOLLAND

Feedback Invariants for Linear Dynamical Systems Over a Principal Ideal Domain

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ABSTRACT

This paper studies the action of the feedback group $\mathbf{F}_{n,m}$ on m-input, n-dimensional reachable linear dynamical systems over a principal ideal domain R. For such a 2-dimensional system Σ a complete set of invariants is given which characterizes the feedback class of Σ. In particular it is characterized, in terms of these invariants, when Σ has the feedback cyclization property. The particular cases $R = \mathbf{Z}$ and $R = \mathbf{R}[X]$ are studied in some detail. Finally, when n is arbitrary, the feedback classification is given for the class of reachable systems

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$\Sigma = (F, G)$ such that G is a matrix with at least $n-1$ invariant factors equal to one.

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1. INTRODUCTION AND NOTATION

Let R be a commutative ring with an identity element. An m-input, n-dimensional linear dynamical system Σ over R is a pair of matrices (F, G), where F is an $n \times n$ matrix and G is an $n \times m$ matrix with elements on R. The system $\Sigma = (F, G)$ is said to be reachable if the columns of the $n \times nm$ matrix.

$$\widetilde{G} = [G, FG, \ldots, F^{n-1}G]$$

generate R^n(i.e., the ideal generated by all the $n \times n$ minors of $\widetilde{G}$ is R).

The feedback group $\mathbf{F}_{n,m}$ is the group, acting on m-input, n-dimensional systems $\Sigma = (F, G)$ over R, generated by the following three types of transformations:

(1) $F \mapsto F' = PFP^{-1}, G \mapsto G' = PG$, for some invertible matrix P. This transformation is a consequence of a change of base in R^n, the state module.

(2) $F \mapsto F, G \mapsto G' = GQ$, for some invertible matrix Q. This transformation is a consequence of a change of base in R^m, the input module.

(3) $F \mapsto F' = F + GK, G \mapsto G$, for some $m \times n$ matrix K, which is called a feedback matrix.

The system Σ' is feedback equivalent to Σ if it is obtained from Σ by one element of $\mathbf{F}_{n,m}$.

The purpose of this paper is the classification of reachable linear dynamical systems by the action of $\mathbf{F}_{n,m}$. When R is a field, the classification is due to P. A. Brunovsky [3], R. E. Kalman [7], and W. A. Wonham and A. S. Morse [9]. In this case the orbits of $\mathbf{F}_{n,m}$ acting on the space of reachable linear dynamical systems correspond bijectively with partitions of the integer n. If $k(\Sigma) = (k_1, k_2, \ldots, k_s)$, where $k_1 \geq k_2 \geq \cdots \geq k_s$ and $\sum_{i=1}^{s} k_i = n$, is the partition associated to Σ, then the integers k_i are called the Kronecker indices of Σ, because they are identical to the classical indices associated with the pencil of matrices $[z\,\mathrm{Id}_n - F \mid G]$.

This paper is devoted to studying the feedback classification problem when R is a principal ideal domain. Before presenting the results of the paper, we recall some basic concepts. We shall say that Σ has the pole assignability property if for each $\lambda_1, \lambda_2, \ldots, \lambda_n$ in R there exists a system $\Sigma' = (F', G')$, feedback equivalent to Σ, such that the characteristic polynomial of $F', \chi(F')$, is $(X - \lambda_1)(X - \lambda_2)\cdots(X - \lambda_n)$. If Σ has the pole assignability property, then Σ is a reachable. In general, the reachability property does not imply the pole assignability property. We shall say that R is a PA_n ring if each reachable system of dimension at most n has the pole property. R is a PA ring if it is a PA_n ring for all n.

For a system with a single input (i.e. $m = 1$) the concepts of reachability and pole assignability are equivalent. The following property is of interest because if allows us to reduce the multiple input case to the single input case. An m-input, n-dimensional reachable system $\Sigma = (F, G)$ has the feedback cyclization property if there exist a system $\Sigma' = (F', G')$, feedback equivalent to Σ, and a vector $\mathbf{w} \in R^m$ such that the system $(F', G'\mathbf{w})$ is reachable. In this case the vector $G'w$ is a cyclic vector for F' (i.e., $G'\mathbf{w}, F'G'\mathbf{w}, \ldots, F'^{n-1}G'\mathbf{w}$ is a basis of R^n). We shall say that R is an FC_n ring if each reachable system of dimension at most n has the feedback cyclization property. R is an FC ring if it is an FC_n ring for all n.

The feedback cyclization property implies the pole assignability property. Any elementary divisor domain, and in particular any principal ideal domain (PID), is a PA ring; see [2, p. 92]. Neither the ring of integers $\mathbf{Z}$ nor the polynomial ring $\mathbf{R}[X]$, where $\mathbf{R}$ is the field of real numbers, is an FC_2 ring. A characterization of FC_2 rings among (almost all) principal ideal domains is given in [4].

The paper is organized as follows. In Section 2, for a m-input, 2-dimensional reachable linear dynamical system Σ over a principal ideal domain we give a complete set of invariants of the class of Σ by the action of $\mathbf{F}_{2,m}$. In this way we obtain the feedback classification for 2-dimensional systems over such a ring. In Section 3, we use this classification to obtain a characterization of a system Σ that has the feedback cyclization property. In particular, we obtain a characterization of FC_2 rings among principal ideal domains which generalizes the quoted results of [4]. This section contains also the study of the cases $R = \mathbf{Z}$ and $R = \mathbf{R}[X]$.

Let $\Sigma = (F, G)$ be an m-input, n-dimensional reachable linear dynamical system over a principal ideal domain such that G is a matrix of rank n with at least $n - 1$ invariant factors equal to one. Note that when $n = 2$ this condition always holds. For $n > 2$ we prove, in Section 4, that Σ has a canonical form depending only on G. This result gives the feedback classification for this class of linear dynamical systems.

In the sequel R will denote a principal ideal domain. For an element d of R we denote by $(R/(d))^*$ the group of units of the quotient ring $(R/(d))$.

2. THE FEEDBACK CLASSIFICATION FOR 2-DIMENSIONAL SYSTEMS

Let $\Sigma = (F, G)$ be an m-input, 2-dimensional linear dynamical system over R. Since R is a PID, there exist invertible matrices P and Q such that the matrix $G' = PGQ$ is the diagonal matrix.

$$G' = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \end{pmatrix},$$

where d_1 divides d_2. Therefore Σ is feedback equivalent to $\Sigma' = (F', G')$ where $F' = PFP^{-1}$. If Σ is reachable, then d_1 is a unit, and hence, by a change of basis in R^m, we can take $d_1 = 1$.

The following result gives, by the action of $\mathbf{F}_{2,m,}$ a (nonunique) canonical form for Σ.

PROPOSITION 2.1. *Let $\Sigma = (F, G)$ be an m-input, 2-dimensional reachable linear dynamical system over R. Then Σ is feedback equivalent to a system $\widehat{\Sigma} = (\widehat{F}, \widehat{G})$ of the form*

$$\widehat{F} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}, \quad \widehat{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix},$$

where d and f are coprime.

Proof. By the above we can suppose that G is the matrix

$$G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix}.$$

Put

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

If K_1 is the $m \times 2$ matrix

$$K_1 = \begin{pmatrix} -f_{11} & -f_{12} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

then Σ is equivalent to $\Sigma' = (F', G')$ with

$$F' = F + GK_1 = \begin{pmatrix} 0 & 0 \\ f_{21} & f_{22} \end{pmatrix}$$

and $\quad G' = G$.

Since Σ is reachable, then Σ' is also reachable. Therefore the columns of the matrix

$$[G', F'G'] = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 & f_{21} & df_{22} & 0 & \cdots & 0 \end{pmatrix}$$

generate R^2. Hence d and f_{21} are coprime. Consequently there exist a and b on R such that $1 = af_{21} + bd$. Consider the following matrices:

$$P = \begin{pmatrix} 1 & af_{22} \\ 0 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & -af_{22}d & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad H = \begin{pmatrix} -af_{22}f_{21} & a^2f_{22}^2f_{21} - af_{22}^2 \\ 0 & -bf_{22} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

and put $K = QH$. Then we have $\widehat{G} = PGQ = G$ and

$$\widehat{F} = PF'P^{-1} + PGK = \begin{pmatrix} 0 & 0 \\ f_{21} & 0 \end{pmatrix}.$$

So $\widehat{\Sigma} = (\widehat{F}, \widehat{G})$ has the desired form, and it is feedback equivalent to Σ. ∎

REMARK 2.2. Note that the element d of the above result is uniquely determined (up to multiplication by a unit) by Σ. In fact, d is a generator of the ideal $\mathcal{U}_2(G)$ generated by all the 2×2 minors of G. However, f is not an invariant of Σ. For this reason we say that $\{f,\, d\}$ is a pair associated to Σ.

Before stating the main result of this section we need the following general result:

LEMMA 2.3. *Let $\Sigma = (F, G)$ and $\Sigma' = (F', G')$ be two reachable m-input, n-dimensional linear dynamical systems over a commutative ring R. Then the following statements are equivalent:*

(i) *Σ' is feedback equivalent to Σ.*

(ii) *There exist an invertible $n \times n$ matrix P, an invertible $m \times m$ matrix Q, and an $m \times n$ matrix K such that $F' = PFP^{-1} + PGK$ and $G' = PGQ$.*

Proof. Let P be an invertible $n \times n$ matrix, Q an invertible $m \times m$ matrix, and K an $m \times n$ matrix. We denote by $\varphi_{(P,Q,K)}(\Sigma)$ the system (F', G') where $F' = PFP^{-1} + PGK$ and $G' = PGQ$.

With this notation we have that the three types of transformations that generate $\mathbf{F}_{n,m}$ are:

(1) $\Sigma \mapsto \varphi_{(P,\,\mathrm{Id}_m,\,0)}(\Sigma)$ where Id_m is the identity matrix and 0 is the zero matrix.

(2) $\Sigma \mapsto \varphi_{(\mathrm{Id}_n,\,Q,\,0)}(\Sigma)$.

(3) $\Sigma \mapsto \varphi_{(\mathrm{Id}_n,\,\mathrm{Id}_m,K)}(\Sigma)$.

We have the following equality:

$$\varphi_{(P',\,Q',\,K')}\big(\varphi_{(P,\,Q,\,K)}(\Sigma)\big) = \varphi_{(P'',Q'',K'')}(\Sigma),$$

where $P'' = P'P, Q'' = QQ'$, and $K'' = KP'^{-1} + QK'$. Therefore, if Σ' is obtained from Σ by composition of transformations of the types (1), (2), and (3), then the above equality proves that there exists (P, Q, K) such that $\varphi_{(P,\,Q,\,K)}(\Sigma) = \Sigma'$. Finally, it is clear that Σ is feedback equivalent to $\varphi_{(P,Q,K)}(\Sigma)$.

■

THEOREM 2.4. *Let $\Sigma = (F, G)$ and $\Sigma' = (F', G')$ be two m-input, 2-dimensional reachable linear dynamical systems over R. Suppose that $\{f, d\}$ and $\{f', d'\}$ are pairs associated to Σ and Σ' respectively. Then Σ is feedback equivalent to Σ' if and only if the following statements hold:*

(i) *The principal ideals dR and $d'R$ are equal.*

(ii) *There exist a unit u of R and an element h of R such that $f' \equiv u\, h^2 f \pmod d$.*

Proof. By Proposition 2.1 we can suppose that

$$F = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix},$$

and

$$F = \begin{pmatrix} 0 & 0 \\ f' & 0 \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d' & \cdots & 0 \end{pmatrix}.$$

Assume that Σ is equivalent to Σ'. By Proposition 2.3, we have $F' = PFP^{-1} + PGK$ and $G' = PGQ$, where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

is invertible, Q is an invertible $m \times m$ matrix, and $K = (k_{ij})$ is an $m \times 2$ matrix.

Since P and Q are invertible, then the equality $G' = PGQ$ implies that $dR = \mathcal{U}_2(G) = \mathcal{U}_2(G') = d'R$, and hence i holds. Since

$$\begin{aligned} F' &= PFP^{-1} + PGK \\ &= u\begin{pmatrix} p_{12}p_{22}f & -p_{12}^2 f \\ p_{22}^2 f & -p_{12}p_{22}f \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} k_{11} & k_{12} \\ dk_{21} & dk_{22} \end{pmatrix}, \end{aligned}$$

it follows that $f' = up_{22}^2 f + p_{21}k_{11} + dp_{22}k_{21}$, where $u = (\det P)^{-1}$. Now the equality $P^{-1}G' = GQ$ implies that $p_{21} \in dR$ and hence $f' = up_{22}^2 f + dr$ with $k \in R$.

Conversely, assume that conditions (i) and (ii) hold. By (i) there exists a unit u_0 of R such that $d = u_0 d'$. Consider the invertible diagonal matrix

$$Q_0 = \begin{pmatrix} 1 & & & & \\ & u_0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then on replacing Σ' by $\varphi_{(\mathrm{Id}_2, Q_0, 0)}(\Sigma')$ we can suppose that $d = d'$.

By (ii), there exists $k \in R$ such that $f' - uh^2 f = dk$. Because f' and d are coprime, it follows that h and d are coprime. Hence there exist elements a and b of R such that $1 = ah + bd^2$. We consider the following matrices:

$$P = \begin{pmatrix} a & d \\ -bd & h \end{pmatrix},$$

$$Q = \begin{pmatrix} h & -d^2 & 0 & \cdots & 0 \\ b & a & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad H = \begin{pmatrix} -dfh & d^2 f \\ u^{-1}k & hf \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

and put $K = QH$. Then

$$\varphi_{(P,Q,K)}(\Sigma) = (F_1, G), \quad \text{with} \quad F_1 = \begin{pmatrix} 0 & 0 \\ u^{-1}f' & 0 \end{pmatrix}.$$

Consequently

$$\varphi_{(P',Q',0)}\big(\varphi_{(P,Q,K)}(\Sigma)\big) = (\Sigma'),$$

where

$$P' = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \quad \text{and} \quad Q' = \begin{pmatrix} 1 & & & & \\ & u^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

This completes the proof. ∎

REMARK 2.5. Let $\Sigma = (F, G)$ be an m-input, 2-dimensional reachable linear dynamical system over R. Suppose that $\{f, d\}$ is a pair associated to Σ. In $(R/(d))^*$ we consider the following equivalence relation: $f \sim f'$ if and only if there exist a unit u of R and an element h of R such that $f' \equiv uh^2 f \pmod{d}$. We denote by $\widetilde{\mathbf{f}}$ the class of f by the above relation.

By Theorem 2.4 we have that $\{\widetilde{\mathbf{f}}, d\}$ is a complete set of invariants of Σ by the action of the feedback group.

3. APPLICATIONS

Let $\Sigma = (F, G)$ be an m-input, 2-dimensional reachable linear dynamical system over R. If $\Sigma = (F, G)$ is feedback equivalent to $\Sigma' = (F', G')$, then G is an equivalent matrix to G'. But the converse is not true in general. For this reason we introduce the following invariant associated to G.

Let G be a $2 \times m$ matrix with coefficients in R such that the content of G is R (i.e., the ideal generated by all elements of G is R). Consider the feedback relation on the set of all reachable systems $\Sigma' = (F', G')$ where G' is equivalent to G. The cardinal of the associated quotient set is denoted by $\omega(G)$. By Theorem 2.4 this positive integer is the cardinal of the quotient set of $(R/(d))^*$ by the equivalence relation introduced in Remark 2.5, where d is a greatest common divisor of the 2×2 minors of G.

Next we shall study for three particular cases the action of the group $\mathbf{F}_{2,m}$ in terms of the function ω.

FC_2 Rings

THEOREM 3.1. *Let $\Sigma = (F, G)$ be an m-input, 2-dimensional reachable linear dynamical system over R. Suppose that $\{f, d\}$ is a pair associated to Σ. Then the following statements are equivalent:*

(i) Σ *has the feedback cyclization property.*

(ii) $f \equiv uh^2 \pmod{d}$, *where u is a unit of R. Equivalently, $\{1, d\}$ is also a pair associated to Σ.*

Proof. By Proposition 2.1 we can suppose that $\Sigma = (F, G)$ with

$$F = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix},$$

with f, d coprime. Suppose that Σ has the feedback cyclization property. Then there exist $\mathbf{w} = (w_1, w_2, \dots, w_m)^t$ in R^m and a matrix $K = k_{ij}$ such that the system $(F + GK, G\mathbf{w})$ is reachable. Hence

$$\begin{aligned} \det[G\mathbf{w}, (F+GK)G\mathbf{w}] &= \det\begin{pmatrix} w_1 & k_{11}w_1 + dk_{12}w_2 \\ dw_2 & fw_1 + dk_{21}w_1 + d^2k_{22}w_2 \end{pmatrix} \\ &= fw_1^2 + d\alpha \end{aligned}$$

is a unit u on R. Consequently $f \equiv uh^2 \pmod{d}$, where $h \equiv w_1^{-1} \pmod{d}$.

Finally it is clear that the system

$$\Sigma = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \end{pmatrix} \right)$$

has the feedback cyclization property. ∎

COROLLARY 3.2. *Let R be a PID. Then the following statements are equivalent:*

(i) *R is an FC_2 ring.*

(ii) *If f and d are two coprime elements of R, then there exists a unit u of R such that uf is a square modulo d.*

(iii) *Let $\Sigma = (F, G)$ and $\Sigma' = (F', G')$ be two m-input, 2-dimensional reachable R-systems. Then Σ is feedback equivalent to Σ' if and only if G is equivalent to G'.*

(iv) *$\omega(G) = 1$ for every $2 \times m$ matrix G with content equal to R.*

REMARK 3.3. In [4, Proposition 4.3] is proved the equivalence between the statements (i) and (ii) of the above result when R/pR has characteristics different from 2 for all irreducibles p of R.

The Case $R = \mathbf{R}[X]$.

LEMMA 3.4. *Let R be a PID, and let $p \in R$ be an irreducible element* *such that $R/(p)$ has characteristic $\neq 2$. Suppose that f is an element of R such that p does not divide f. Then the following statements are equivalent:*

(i) *f is a square modulo p.*
(ii) *f is a square modulo p^r for each integer r.*
(iii) *f is a square modulo p^r for some integer r.*

Proof. It is sufficient to prove (i) $\Rightarrow$ (ii). Let us apply induction on r. Suppose that $f \equiv h^2 (\text{mod } p^{r-1})$. Then there exists $g \in R$ such that $f = h^2 + gp^{r-1}$. Now it is easy to verify that $f \equiv (h + 2^{-1}h^{-1}gp^{r-1})^2 \, (\text{mod } p^r)$. ■

PROPOSITION 3.5. *Let $d(X) \in \mathbf{R}[X]$, and let $\lambda_1, \lambda_2, \ldots, \lambda_t$ be the roots of $d(X)$ in $\mathbf{R}$. Suppose that f and f' are two elements of $\mathbf{R}[X]$ invertible in $\mathbf{R}[X]/(d(X))$. Then the following statements are equivalent:*

(i) *$f'(X) \equiv uh(X)^2 f(X) (\text{mod}\, d(X))$, where u is a unit in $\mathbf{R}[X]$.*
(ii) *Either* sign $f'(\lambda_i)$ = sign $f(\lambda_i)$ *for each i or* sign $f'(\lambda_i)$ = $-$sign $f(\lambda_i)$ *for each i.*

Proof. Let $d(X) = (X-\lambda_1)^{r_1} (X-\lambda_2)^{r_2} \cdots (X-\lambda_t)^{r_t} p_1(X)^{s_1} p_2(X)^{s_2} \cdots p_h(X)^{s_h}$ be an irreducible discomposition of $d(X)$ where the degree of p_i is two for $i = 1, 2, \ldots, h$. By the Chinese remainder theorem we have

$$(\mathbf{R}[X]/(d(X)))^* \cong \prod_{i=1}^{t} \left(\frac{\mathbf{R}[X]}{(X-\lambda_i)^{r_i}} \right)^* \times \prod_{j=1}^{h} \left(\frac{\mathbf{R}[X]}{(p_j(X)^{s_j})} \right)^*.$$

Assume (i). Then $f'(X) \equiv uh(X)^2 f(X) \, (\text{mod } (X-\lambda_i)^{r_i})$ for each i. By Lemma 3.4 it follows that $f'(X) \equiv uh(X)^2 f(X) \, (\text{mod } (X-\lambda_i))$ for each i. Therefore $f'(\lambda_i) = uh(\lambda_i)^2 f(\lambda_i)$ for each i. Since u is a unit on $\mathbf{R}$, it follows that (ii) holds.

Assume (ii). Taking either $u = 1$ or $u = -1$, we have that $f'(\lambda)/uf(\lambda_i)$ is positive for $i = 1, 2, \ldots, t$. Consequently $f'(X)/uf(X)$ is a square modulo $(X-\lambda_i)$ for each i, and hence, by Lemma 3.4, $f'(X)/uf(X)$ is a square modulo $(X-\lambda_i)^{r_i}$ for each i. Since $\mathbf{R}[X]/(p_j(X))$ is isomorphic to $\mathbf{C}$, it follows by Lemma 3.4 that $f'(X)/uf(X)$ is a square modulo $p_j(X)^{s_j}$ for each j. By the Chinese remainder theorem it follows that $f'(X)/uf(X)$ is a square modulo $d(X)$. ■

COROLLARY 3.6. *Let*

$$G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & d(X) & \cdots & 0 \end{pmatrix} \quad \textit{with} \quad d(X) \in \mathbf{R}[X].$$

Suppose that $d(X)$ has t different roots on $\mathbf{R}$. *Then*

$$\omega(G) = \begin{cases} 1 & \text{if} \quad t = 0, \\ 2^{t-1} & \text{if} \quad t > 0. \end{cases}$$

In particular, every reachable system $\Sigma = (F, G)$ has the feedback cyclization property if and only if $d(X)$ has at most one real root.

The Case $R = \mathbf{Z}$.

Let $d \in \mathbf{Z}$. We shall denote by $\delta(d)\,[\partial(d)]$ the number of different solutions of the equation $X^2 - 1 = 0\,[X^2 + 1 = 0]$ in $\mathbf{Z}/(d)$.

LEMMA 3.7. *Let p be an irreducible integer.*

(i) *If $p \neq 2$ then $\delta(p^r) = 2$ for $r \geq 1$.*
(ii) *$\delta(2) = 1, \delta(2^2) = 2$, and $\delta(2^r) = 4$ for $r \geq 3$.*
(iii) *If $p \equiv 3 \pmod 4$ then $\partial(p^r) = 0$ for $r \geq 1$.*
(iv) *If $p \equiv 1 \pmod 4$ then $\partial(p^r) = 2$ for $r \geq 1$.*
(v) *$\partial(2) = 1$ and $\partial(2^r) = 0$ for $r \geq 2$.*

Proof. Suppose that $p \neq 2$. In [1, p. 77] it is proved that if the equation $X^2 = a$ has a solution in $\mathbf{Z}/(p)$, then $X^2 = a$ has two different solutions in $\mathbf{Z}/(p^r)$. Consequently (i) holds. Statements (iii) and (iv) hold because (see [1, p. 109]) the equation $X^2 \equiv -1 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 4$.

Next we shall consider the case $p = 2$. It is clear that $\delta(2) = 1, \delta(2^2) = 2$, and $\delta(2^3) = 4$. Let $r \geq 4$, and suppose that the solutions of $X^2 = -1$ in $\mathbf{Z}/(2^{r-1})$ are $\{1, 1 + 2^{r-2}, -1, -1 + 2^{r-2}\}$. If $\lambda^2 \equiv 1 \pmod{2^r}$, then there exist $a \in \{1, -1\}$ and $b \in \{0, 1\}$ such that $\lambda \equiv a + b\,2^{r-2} \pmod{2^{r-1}}$. Therefore $\lambda \equiv a + b\,2^{r-2} + c\,2^{r-1} \pmod{2^r}$ where $c \in \{0, 1\}$. It follows that $1 = \lambda^2 = a^2 + ab2^{r-1} \equiv 1 + ab2^{r-1} \pmod{2^r}$ and hence $b = 0$. Consequently the solutions of $X^2 = -1$ in $\mathbf{Z}/(2^r)$ are $\{1, 1 + 2^{r-1}, -1, -1 + 2^{r-1}\}$.

Finally, (v) is clear because $X^2 = -1$ has no solution in $\mathbf{Z}/(2^2)$. ∎

By the Chinese remainder theorem we have the following result.

COROLLARY 3.8. *Suppose that $d = 2^r p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$, where all p_i are odd primes. Then*

(i) *we have*

$$\delta(d) = \begin{cases} 2^t & \text{if} \quad r = 0, 1, \\ 2^{t+1} & \text{if} \quad r = 2, \\ 2^{t+2} & \text{if} \quad r \geq 3; \end{cases}$$

(ii) $\partial(d) \geq 1$ *if and only if* $r \leq 1$ *and* $p_i \equiv 1 \pmod 4$ *for each* i.

PROPOSITION 3.9. *Let*

$$G = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & d & \dots & 0 \end{pmatrix} \qquad \textit{with} \quad d \in \mathbf{Z}.$$

Then we have

$$\omega(G) = \begin{cases} \delta(d) & \text{if} \quad \partial(d) > 0, \\ \delta(d)/2 & \text{if} \quad \partial(d) = 0. \end{cases}$$

In particular, $\omega(G) = 1$ *(i.e., every reachable system* $\Sigma = (F, G)$ *has the feedback cyclization property) if and only if either* $d = 2$ *or* $d = 4$ *or* $d = 2^r p^s$ *where* $r \in \{0, 1\}$ *and* p *is a prime such that* $p \equiv 3 \pmod 4$.

Proof. Let $\widetilde{\mathbf{f}}$ be the equivalence class of $f \in (\mathbf{Z}/(d))^*$ by the equivalence relation $\sim$ introduced in Remark 2.5. Since the cardinal of $\widetilde{\mathbf{f}}$ is equal to the cardinal of $\widetilde{1}$, it follows that

$$\phi(d) = \omega(G) \cdot \sharp(\widetilde{1})$$

where $\phi(d)$ is Euler's phi function and $\sharp(\widetilde{1})$ is the cardinality of $\widetilde{1}$.

Next we obtain $\sharp(\widetilde{1})$. First note that $\widetilde{1}$ is the union of sets $A = \{ h^2/h \in (\mathbf{Z}/(d))^*\}$ and $B = \{-h^2/h \in (\mathbf{Z}/(d))^*\}$. Moreover, if $\partial(d) > 0$ then $A \cap B = A = B$, and if $\partial(d) = 0$ then $A \cap B$ is empty. Since $\sharp A = \sharp B = \phi(d)/\delta(d)$, it follows that

$$\sharp(\widetilde{1}) = \begin{cases} \phi(d)/\delta(d) & \text{if} \quad \partial(d) > 0, \\ 2(\phi(d)/\delta(d)) & \text{if} \quad \partial(d) = 0, \end{cases}$$

which completes the proof. ∎

4. A CLASSIFICATION THEOREM FOR m-DIMENSIONAL SYSTEMS

Let R be a PID, and let n be an integer with $n \geq 3$. In this section we study the feedback class of an m-input, n-dimensional reachable linear dynamical R-system $\Sigma = (F, G)$ where G is equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d & \cdots & 0 \end{pmatrix}.$$

In particular note that $m \geq n$.

LEMMA 4.1. *Let $\Sigma = (F, G)$ be the m-input, n-dimensional linear dynamical R-system given by*

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_1 & f_2 & \cdots & f_k & 0 & \cdots & 0 \end{pmatrix}$$

and

$$G = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d & \cdots & 0 \end{pmatrix},$$

where $k < n$. If f is a greatest common divisor of $f_1, f_2, \dots, f_k$, then Σ is equivalent to $\Sigma' = (F', G')$, where

$$F' = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$G' = G = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d & \cdots & 0 \end{pmatrix}.$$

Proof. Let t_{k-1} be the greatest common divisor of f_{k-1}, and f_k, and put $f_{k-1} = t_{k-1} f'_{k-1}$ and $f_k = t_{k-1} f'_k$. There exist elements a_{k-1} and b_k of R such that $1 = a_{k-1} f'_{k-1} + b_k f'_k$. Consider the invertible block matrices

$$P_k = \begin{pmatrix} \mathrm{Id}_{k-2} & & \\ & A & \\ & & \mathrm{Id}_{n-k} \end{pmatrix} \quad \text{and} \quad Q_k = \begin{pmatrix} \mathrm{Id}_{k-2} & & \\ & A^{-1} & \\ & & \mathrm{Id}_{m-k} \end{pmatrix},$$

where

$$A = \begin{pmatrix} f'_{k-1} & f'_k \\ b_k & -a_{k-1} \end{pmatrix},$$

and Id_r, denotes the identity matrix of order r. Then we have

$$P_k F P_k^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ f_1 & f_2 & \cdots & f_{k-2} & t_{k-1} & 0 & \cdots & 0 \end{pmatrix},$$

and $P_k G Q_k = G$. The result follows by iteration of this process. ■

THEOREM 4.2. *Let $\Sigma = (F, G)$ be a m-input, n-dimensional reachable linear dynamical system over R where $n \geq 3$ and G is equivalent to the matrix*

$$\widehat{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d & \cdots & 0 \end{pmatrix}.$$

Then Σ is feedback equivalent to $\widehat{\Sigma} = (\widehat{F}, \widehat{G})$, where

$$\widehat{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. First note that we can suppose that G is the matrix $\widehat{G}$. Let $F = (f_{ij})$. If K_1 is the $m \times n$ matrix

$$K_1 = \begin{pmatrix} -f_{11} & -f_{12} & \cdots & -f_{1n} \\ \vdots & \vdots & \vdots & \\ -f_{n-1,1} & -f_{n-1,2} & \cdots & -f_{n-1,n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then Σ is equivalent to $\Sigma' = (F', G')$ with $G' = G$ and

$$F' = F + GK_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ f_1 & f_2 & \cdots & f_n \end{pmatrix},$$

where $f_k = f_{nk}$ for $k = 1, \ldots, n$. Since Σ is reachable, then Σ' is also reachable. Therefore the ideal of R generated by $d, f_1, f_2, \ldots, f_{n-1}$ is R. Consequently there exist elements $b, a_1, a_2, \ldots, a_{n-1}$ in R such that

$$1 = bd + a_1 f_1 + a_2 f_2 + \cdots + a_{n-1} f_{n-1}.$$

We consider the invertible $n \times n$ matrix

$$P = \begin{pmatrix} 1 & \cdots & 0 & a_1 f_n \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_{n-1} f_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the invertible $m \times m$ matrix

$$Q = \begin{pmatrix} 1 & \cdots & 0 & -a_1 f_n d & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & -a_{n-1} f_n d & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we have that $PGQ = G$ and

$$PF'P^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,n-} & \alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots & \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \cdots & \alpha_{n-1,n-1} & \alpha_{n-1,n} \\ f_1 & f_2 & \cdots & f_{n-1} & bdf_n \end{pmatrix},$$

where $\alpha_{i,j} \in R$. If K_2 is the $m \times n$ matrix

$$K_2 = \begin{pmatrix} -\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1,n-1} & -\alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots & \\ -\alpha_{n-1,1} & -\alpha_{n-1,2} & \cdots & -\alpha_{n-1,n-1} & -\alpha_{n-1,n} \\ 0 & 0 & \cdots & 0 & -bf_n \end{pmatrix},$$

it follows that Σ is equivalent to $\Sigma'' = (F'', G'')$, where $G'' = G$ and

$$F'' = PF'P^{-1} + G''K_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ f_1 & f_2 & \cdots & f_{n-1} & 0 \end{pmatrix}.$$

By Lemma 4.1, Σ is equivalent to $\Sigma'' = (F''', G)$ with

$$F''' = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ f & 0 & \cdots & 0 \end{pmatrix},$$

where f is the greatest common divisor of $f_1, f_2, \ldots, f_{n-1}$. Since Σ is reachable, it follows that f and d are coprime. Consequently there exist elements u and v of R such that $1 = uf + vd$. We consider the following invertible block matrices:

$$P = \begin{pmatrix} A & \\ & \mathrm{Id}_{n-2} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} A^{-1} & \\ & \mathrm{Id}_{m-2} \end{pmatrix},$$

where

$$A = \begin{pmatrix} f & -d \\ v & u \end{pmatrix}.$$

Put $K = QH$, where H is the $m \times n$ matrix

$$H = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ v & -f & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\varphi_{(P,Q,K)}(\Sigma''') = \widehat{\Sigma} = (\widehat{F}, \widehat{G})$. ■

REMARK 4.3. Note that G is equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d & \cdots & 0 \end{pmatrix}$$

if and only if the ideal generated by all the $(n-1)$-minors of G is R.

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